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REGULAR AND SINGULAR PERTURBATION SOLUTIONS FOR BENDING AND TORSION OF BEAMS

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Abstract—For the bending of a beam under axial forces and for the warping of a shaft in torsion, a perturbation solution of the governing differential equation is derived. The beam or shaft may have variable cross sections, variable loading and general end conditions. For both cases—the singular and regular one—the perturbation solutions are explicitly carried out up to (and including) the first perturbation term. For the singular case in addition a solution uniformly valid over the entire beams length is constructed. Two examples illustrating the application are given at the end of this paper.

1. INTRODUCTION

IT IS well known in structural mechanics that the bending of a beam under axial force and the torsion of the beam with constrained warping of the cross sections are governed by the same fourth order ordinary differential equation. When the cross sections or the applied axial forces and torques are constant along the span length, the differential equation has constant coefficients and an exact solution is readily obtainable. On the other hand, if either the cross-section or the applied force or torque change along the length, the coefficients of the equation are then functions of the length coordinate and solutions are difficult to obtain. In such a case methods based on energy principles, finite differences, method of finite elements or other numerical schemes are used to find approximate solutions.

When the differential equation is analyzed by one of the numerical methods mentioned above, an algebraic system of linear equations has to be solved. In addition, when the finite difference method is applied, there is the need to study the stability and convergence of the solutions which may be rather difficult.

It is then clear we need another method to solve the stated problems of structural mechanics without encountering such numerical difficulties when the coefficient of the highest order term in the differential equation is small. This happens to be the case in warping-torsion for closed thin-walled cross-sections. The same is true for cables with small bending rigidity under high tension.

The differential equations of the bending or twisting of beams are given by [1, 2]

$$[EI(x)w(x)'']''-[N(x)w(x)']' = q(x); \qquad 0 \le x \le L; \tag{1.1a}$$

$$[EJ(x)\psi(x)'']''-[GK(x)\psi(x)']' = t(x); \qquad 0 \le x \le L, \tag{1.1b}$$

together with the usual boundary conditions at $x = 0$ and $x = L$. The symbols in (1.1) represent:

$EI(x)$	bending rigidity
$EJ(x)$	warping factor
$N(x)$	axial force (tension positive)

$GK(x)$	torsional rigidity
$w(x)$	transverse displacement
$\psi(x)$	angle of twist
$q(x)$	specific loading (transverse force per unit length)
$t(x)$	specific torsional moment
x	coordinate measured along the beam
$(\ )'$	derivative with respect to x.

For the case of bending, the bending moment M and shear force Q are related to w by

$$M = -EIw'', \qquad Q = -(EIw'')', \tag{1.2a}$$

whereas for the torsional problem, the Saint Venant torque T_s, the bimoment M_ω and the warping moment T_ω are related to ψ by

$$T_s = GK\psi'', \qquad M_\omega = -EJ\psi'', \qquad T_\omega = -(EJ\psi'')'. \tag{1.2b}$$

The fact that the different physical behaviors lead to the same differential equations makes it possible to discuss the two problems simultaneously.

The integration of the equation (1.1) in closed form is hardly possible for variable coefficients. When the rigidity or the axial force are general functions of x exact solutions are not known. On the other hand, if one or the other of the coefficients of the left hand side of (1.1) is much smaller than the second, analytical solutions are obtainable from a perturbational approach with good accuracy. If, for example

$$N_0L^2 \ll EI_0; \qquad GK_0L^2 \ll EJ_0, \tag{1.3}$$

where the subscript 0 designates a typical characteristic value of the corresponding function, the highest derivatives in (1.1) are of greater influence and one is led to the case of regular perturbation. In the inverse case, namely

$$N_0L^2 \gg EI_0; \qquad GK_0L^2 \gg EJ_0, \tag{1.4}$$

the lower order terms are more important and the perturbation becomes singular at $x = 0$ and $x = L$. Therefore one has to distinguish between a solution far away from the boundaries (a so-called *outer* solution) and a *boundary-layer* solution (inner solution) which satisfies the boundary conditions of the higher order differential equation (1.1).

On physical ground the coefficients of the equation (1) satisfy the conditions

$$EI > 0; \qquad EJ > 0; \qquad GK > 0; \qquad 0 \leq x \leq L. \tag{1.5}$$

For N, one has the possibilities $N \lesseqgtr 0$ in the considered interval $[0, L]$. For the singular case we consider only the case $N > 0$, because the case $N < 0$ belongs to the class where buckling will occur prior to the satisfaction of conditions (1.4). In the regular case a change of the sign of N is permitted in the perturbation approach.

We shall discuss first the regular perturbation solution of (1.1) when the condition (1.3) is satisfied. The singular perturbation solution for the case defined by (1.4) is presented in Section 3, and in the last section we illustrate the applications for two problems in structural engineering.

2. REGULAR PERTURBATION SOLUTION

The transformations

$$
\left.\begin{aligned}
x &= \xi L; \\
\Phi &= \begin{cases} (EI_0/PL^4)w; & \text{bending} \\ (EJ_0/TL^4)\psi; & \text{twisting} \end{cases} \\
q &= Qp; && \text{bending}\dagger \\
t &= Tp; && \text{twisting}\dagger \\
N &= N_0 b; && \text{bending} \\
GK &= GK_0 b; && \text{twisting}
\end{aligned}\right. \tag{2.1}
$$

change the differential equation (1.1) to the dimensionless form

$$
\left.\begin{aligned}
&[a(\xi)\Phi'']'' - \varepsilon[b(\xi)\Phi']' = p(\xi); \qquad 0 \le \xi \le 1; \\
&a(\xi) = \begin{cases} EI/EI_0; \\ EJ/EJ_0; \end{cases} \quad b(\xi) = \begin{cases} N/N_0; & \text{bending} \\ GK/GK_0; & \text{twisting} \end{cases} \\
&\varepsilon = \begin{cases} (N_0 L^2)/(EI_0); & \text{bending} \\ (GK_0 L^2)/(EJ_0); & \text{twisting} \end{cases}
\end{aligned}\right\} \tag{2.2}
$$

In the above the prime symbol indicates differentiation with respect to ξ, and because of the restrictions as specified in (1.5) one finds

$$
\begin{aligned} 0 < a(\xi) \le 1; \\ 0 \le b(\xi) \le 1; \end{aligned} \qquad 0 \le \xi \le 1. \tag{2.3}
$$

In what follows we assume as in (1.2),

$$
|\varepsilon| \ll 1 \tag{2.4}
$$

and try to solve the differential equation (2.2) in a form of powers of ε, namely

$$
\Phi = \Phi_0 + \varepsilon\Phi_1 + \dots. \tag{2.5}
$$

Substitution of this into (2.2), comparing the terms of equal powers in ε leads to a system of differential equations, the solutions of which are given as

$$
\Phi_0 = C_0 + D_0\xi + B_0\int_0^{\xi}\frac{(\xi-\lambda)}{a(\lambda)}\,d\lambda + A_0\int_0^{\xi}\lambda\frac{(\xi-\lambda)}{a(\lambda)}\,d\lambda + \int_0^{\xi}\left[\frac{(\xi-\lambda)}{a(\lambda)}\int_0^{\lambda}(\lambda-\phi)p(\phi)\,d\phi\right]d\lambda, \tag{2.6}
$$

† We best choose P and T such that

$$
QL = \int_0^L |p(x)|\,dx, \qquad TL = \int_0^L |t(x)|\,dx.
$$

$$\phi_1 = C_1+D_1\xi+B_1\int_0^\xi \frac{d\lambda(\xi-\lambda)}{a(\lambda)}+A_1\int_0^\xi \frac{\lambda\, d\lambda}{a(\lambda)}(\xi-\lambda)+D_0\int_0^\xi \frac{\xi-\lambda}{a(\lambda)}\int_0^\lambda b(\phi)\, d\phi\, d\lambda$$
$$+B_0\int_0^\xi \frac{\xi-\lambda}{a(\lambda)}\int_0^\lambda b(\phi)\int_0^\phi \frac{1}{a(\zeta)}\, d\zeta\, d\phi\, d\lambda+A_0\int_0^\xi \frac{\xi-\lambda}{a(\lambda)}\int_0^\lambda b(\phi)\int_0^\phi \frac{\zeta\, d\zeta}{a(\zeta)}\, d\phi\, d\lambda$$
$$+\int_0^\xi \frac{\xi-\lambda}{a(\lambda)}\int_0^\lambda b(\phi)\int_0^\phi \frac{1}{a(\zeta)}\int_0^\zeta (\zeta-\omega)p(\omega)\, d\omega\, d\zeta\, d\phi\, d\lambda. \tag{2.7}$$

The unknown coefficients of the zeroth order, A_0, B_0, C_0 and D_0 are to be determined by the original boundary conditions and those in first order solutions are fixed by homogeneous boundary conditions. Higher order perturbation solutions can be determined analogously.

3. SINGULAR PERTURBATION

Singular perturbation problems arise when one is seeking an approximation of the solution of (1.1) where the effect of bending or warping in (1.1) is relatively small in comparison to the tension or to the Saint Venant torsion, respectively. For special boundary conditions, constant cross sections and constant normal force this case is discussed for the bending member in [3]. The associated vibrating problem has been solved in [4]. Here we extend the analysis to variable cross sections, valid for general boundary conditions.

Introducing the transformations (2.1) where we replace Φ by

$$\chi = \begin{cases} [N_0/PL^2)]w; & \text{bending} \\ [(GK_0)/(TL^2)]\psi; & \text{twisting,} \end{cases} \tag{3.1}$$

equation (1.1) assumes the dimensionless form

$$\eta^2[a(\xi)\chi'']''-[b(\xi)\chi']' = p(\xi); \qquad 0 \le \xi \le 1, \tag{3.2}$$

where

$$a(\xi) = \begin{cases} EI_0/EI; \\ EJ_0/EJ; \end{cases} \quad b(\xi) = \begin{cases} N_0/N; \\ GK_0/GK; \end{cases} \quad \eta^2 = \begin{cases} EI_0/(N_0L^2); & \text{bending} \\ EJ_0/(GK_0L^2); & \text{twisting.} \end{cases} \tag{3.3}$$

In view of the equations (1.4) and (2.3), the coefficients $a(\xi)$ and $b(\xi)$ are again bounded between 0 and 1.

We now solve (3.2) under the restriction that $\eta^2 \ll 1$.

(a) *Outer solution*

Following Ref. [3], we try to find an outer expansion in the form

$$\chi(\xi, \eta) = h_0+\eta h_1+\dots. \tag{3.4}$$

Substituting (3.4) into (3.2) and comparing terms of the same power in η gives rise to the system of differential equations

$$\begin{aligned} -[b(\xi)h_0']' &= p(\xi); \\ [b(\xi)h_1']' &= 0; \\ [b(\xi)h_2']' &= [a(\xi)h_0'']''. \\ &\vdots \end{aligned} \tag{3.5}$$

We restrict our considerations to the zeroth and first order solutions with

$$h_0 = B_0 + A_0 \int_0^{\xi} \frac{d\lambda}{b(\lambda)} - \int_0^{\xi} \frac{d\lambda}{b(\lambda)} \int_0^{\lambda} p(\phi)\, d\phi ; \tag{3.6a}$$

$$h_1 = B_1 + A_1 \int_0^{\xi} \frac{d\lambda}{b(\lambda)}, \tag{3.6b}$$

where $A_i, B_i, (i = 0, 1)$ are arbitrary constants, which are determined by matching the outer solution with the following boundary layer solutions.

(b) *Boundary layer expansion near* $\xi = 0$

Near the boundaries the influence of the higher order derivatives in (3.2) becomes greater in comparison to the lower order terms. In order to balance their influence, we introduce the boundary layer coordinate near $\xi = 0$

$$\tilde{\xi} = \xi/\eta \tag{3.7}$$

which transforms (3.2) into

$$\frac{1}{\eta^2}\{[a(\xi)\chi''(\tilde{\xi})]'' - [b(\xi)\chi'(\tilde{\xi})]'\} = p(\tilde{\xi}). \tag{3.8}$$

Introducing the power series expansions for the coefficients a and b and p

$$\begin{aligned} a(\xi) &= a_0^{(0)} + a_1^{(0)}\xi + \ldots = a_0^{(0)} + \eta\tilde{\xi}a_1^{(0)} + \ldots ; \\ b(\xi) &= b_0^{(0)} + b_1^{(0)}\xi + \ldots = b_0^{(0)} + \eta\tilde{\xi}b_1^{(0)} + \ldots , \\ p(\xi) &= p_0^{(0)} + \eta\tilde{\xi}p_1^{(0)} + \ldots , \end{aligned} \tag{3.9}$$

assuming a boundary layer expansion of the form

$$\chi(\tilde{\xi}) = g_0(\tilde{\xi})\eta + g_1(\tilde{\xi})\eta^2 + \ldots \tag{3.10}$$

and substituting from (3.9) and (3.10) into (3.8) in which $p(\tilde{\xi})$ is expanded into a power series about $\tilde{\xi} = 0$, and collecting terms of the same powers in η, one obtains the differential equations

$$[a_0^{(0)}g_0'']'' - [b_0^{(0)}g_0']' = 0 \tag{3.11}$$

$$[a_0^{(0)}g_1'']'' - [b_0^{(0)}g_1'] = p_0^{(0)} - [(a_1^{(0)}\tilde{\xi}g_0'')'' - (b_1^{(0)}\tilde{\xi}g_0')']. \tag{3.12}$$

$$\vdots$$

For a first order approximation we need only to solve equation (3.11). Its solution is

$$g_0(\tilde{\xi}) = C_0 + D_0\tilde{\xi} + E_0 \exp\left(-\frac{b_0^{(0)}}{a_0^{(0)}}\tilde{\xi}\right) \tag{3.13}$$

where we already have discarded exponentially growing terms which do not allow matching with the outer solution. Two of the three constants in (3.13) are determined through the boundary conditions at $\xi = 0$.

(c) *Boundary layer expansion near* $\xi = 1$

Physically, there is no difference between the boundary layer near the left and right end. We therefore only have to translate to the neighborhood of $\xi = 1$ of what was said for the neighborhood of $\xi = 0$ [see (b)]. We therefore write

$$\hat{\xi} = (1-\xi)/\eta \tag{3.14}$$

so that $\chi(\hat{\xi})$ is of the form

$$\chi(\hat{\xi}) = \eta f_0(\hat{\xi}) + \eta^2 f_1(\hat{\xi}) + \dots. \tag{3.15}$$

Expanding the coefficients a and b near $\xi = 1$ we obtain

$$\begin{aligned} a(\xi) &= a_0^{(1)} + a_1^{(1)}(\xi - 1) + \dots \\ &= a_0^{(1)} - a_1^{(1)}\eta\hat{\xi} + \dots; \\ b(\xi) &= b_0^{(1)} - b_0^{(1)}\eta\hat{\xi} + \dots. \end{aligned} \tag{3.16}$$

In exactly the same way we can expand also the loading p:

$$p(\xi) = p_0^{(1)} - p_1^{(1)}\eta\hat{\xi} + \dots. \tag{3.17}$$

Substituting (3.14)–(3.17) into (3.2) and comparing terms of the same order in η leads to the equations

$$\begin{aligned} &a_0^{(1)} f_0^{IV} - b_0^{(1)} f_0'' = 0; \\ &a_0^{(1)} f_1^{IV} - b_0^{(1)} f_1'' = p_0^{(1)} + (a_1^{(1)} \cdot \hat{\xi} f_0'')'' - (b_1^{(1)} \cdot \hat{\xi} f_0')', \end{aligned} \tag{3.18}$$

the first of which has the solution

$$f_0(\hat{\xi}) = G_0 + H_0\hat{\xi} + K_0 \exp\left(-\frac{b_0^{(1)}}{a_0^{(1)}}\hat{\xi}\right), \tag{3.19}$$

where the term involving $\exp(+b_0^{(1)}\hat{\xi}/a_0^{(1)})$ has been dropped owing to the fact that exponential growth allows no matching as $\hat{\xi} \to +\infty$.

(d) *Matching procedure*

Matching (3.13) with the outer solution requires a coordinate transformation by introducing an intermediate coordinate ξ^*

$$\xi^* = \xi/\gamma(\eta), \tag{3.20}$$

such that it follows from $\eta \to 0$

$$\gamma/\eta \to \infty; \qquad \hat{\xi} = (\gamma/\eta)\xi^* \to \infty; \qquad \xi = \gamma\xi^* \to 0. \tag{3.21}$$

(These requirements are for example satisfied for $\gamma = \eta^a, 0 < a < 1$.) The meaning of (3.21) is that for fixed and finite ξ^*, the boundary layer coordinate $\hat{\xi}$ and the outer expansion coordinate ξ both tend simultaneously to the appropriate limits. Therefore, in terms of ξ^*, both expansions are valid for $\eta \to 0$, and we have

$$\lim_{\eta \to 0(\xi^* \text{fixed})} (h_0 + \eta h_1 + \dots - g_0\eta - g_1\eta^2 - \dots) = 0.$$

Expanding h_0 and h_1 in a power series near $\xi = 0$ and using (3.13) and (3.21) we obtain

$$\begin{aligned} \lim_{\eta \to 0(\xi^* \text{fixed})} \Big\{ &B_0 + \frac{A_0}{b(0)}\gamma\xi^* + \dots + \eta B_1 + \frac{A_1}{b(0)}\eta\gamma\xi^* + \dots \\ &-\eta C_0 - D_0\gamma\xi^* - \eta E_0 \exp\left(-\frac{b_0^{(0)}}{a_0^{(0)}}\frac{\gamma}{\eta}\xi^*\right)\Big\} = 0. \end{aligned} \tag{3.22}$$

This shows that

$$B_0 = 0; \qquad \frac{A_0}{b(0)} = D_0; \qquad B_1 = C_0. \tag{3.23}$$

Substituting (3.23) in (3.13), we find the first term of the boundary layer solution (3.10) is

$$\chi(\tilde{\xi}) = \eta C_0 + \frac{A_0}{b_0^{(0)}}\tilde{\xi} + E_0 \exp\left(-\frac{b_0^{(0)}}{a_0^{(0)}}\tilde{\xi}\right) + \dots, \tag{3.24}$$

where of the three unknown constants, C_0 and E_0 are determinable through the boundary conditions at $\xi = 0$.

An intermediate limit analogous to the one given in (3.20) is given through the transformation

$$\xi^+ = (\xi - 1)/\zeta(\eta), \tag{3.25}$$

which satisfies the conditions that if ξ is fixed and $\eta \to 0$ then

$$\tilde{\xi} = \frac{\zeta}{\eta}\xi^+ \to +\infty \quad \text{and} \quad \xi = \zeta\xi^+ + 1 \to 1.$$

Therefore the condition for matching the boundary layer solution (3.15) with the outer solution (3.4) is [obtainable in the same way as (3.22) was obtained]

$$\lim_{\eta \to 0(\xi^* \text{fixed})} \left\{ A_0 \int_0^1 \frac{d\lambda}{b(\lambda)} - \int_0^1 \frac{d\lambda}{b(\lambda)} \int_0^\lambda p(\phi)\, d\phi + \left[\frac{A_0}{b(1)} - \frac{1}{b(1)}\int_0^1 p(\lambda)\, d\lambda\right]\zeta\xi^+ \right.$$
$$\left. + \eta C_0 + \eta A_1 \int_0^1 \frac{d\lambda}{b(\lambda)} + \eta \frac{A_1}{b(1)}\zeta\xi^+ + \dots - \eta G_0 - H_0\zeta\xi^+ - \eta K_0 \exp\left(-\frac{b_0^{(1)}}{a_0^{(1)}}\frac{\zeta}{\eta}\xi^+\right)\right\} = 0, \tag{3.26}$$

where use has been made of the results given in (3.23) and (3.25) and a power series expansion of the outer solution about $\xi = 1$ has been used. From the above limit one obtains

$$A_0 = \frac{\int_0^1 [d\lambda/b(\lambda)] \int_0^\lambda p(\phi)\, d\phi}{\int_0^1 [d\lambda/b(\lambda)]}; \tag{3.27}$$

$$A_1 = \frac{G_0 - C_0}{\int_0^1 [d\lambda/b(\lambda)]}; \tag{3.28}$$

$$H_0 = +\frac{\int_0^1 [d\lambda/b(\lambda)] \int_0^\lambda p(\phi)\, d\phi}{b_0^{(1)} \int_0^1 [d\lambda/b(\lambda)]} - \frac{1}{b_0^{(1)}}\int_0^1 p(\lambda)\, d\lambda, \tag{3.29}$$

where use has been made that $b_0^{(1)} = b(1)$ as seen from (3.16). The remaining unknown constants G_0 and K_0 are to be determined by the end conditions at $\xi = 1$.

(e) *The final solution*

To summarize, the outer solution for $0 < \xi < 1$ and the two boundary layer solutions are:

(a) Near the end $\xi = 0$:

$$\chi(\xi) = \eta \left\{ C_0 + \frac{\int_0^1 [d\lambda/b(\lambda)] \int_0^\lambda p(\phi)\, d\phi}{b_0^{(0)} \int_0^1 [d\lambda/b(\lambda)]}\tilde{\xi} + E_0 \exp\left(-\frac{b_0^{(0)}}{a_0^{(0)}}\tilde{\xi}\right)\right\}, \tag{3.30}$$

where $\tilde{\xi} = \xi/\eta$.

(b) Between the ends, $0 < \xi < 1$:

$$\chi(\xi) = \frac{\int_0^1 [\mathrm{d}\lambda/b(\lambda)] \int_0^\lambda p(\phi)\,\mathrm{d}\phi}{\int_0^1 [\mathrm{d}\lambda/b(\lambda)]} \int_0^\xi \frac{\mathrm{d}\lambda}{b(\lambda)} - \int_0^\xi \frac{\mathrm{d}\lambda}{b(\lambda)} \int_0^\lambda p(\phi)\,\mathrm{d}\phi$$
$$+\eta\left[C_0 - \frac{C_0 - G_0}{\int_0^1 [\mathrm{d}\lambda/b(\lambda)]} \cdot \int_0^\xi \frac{\mathrm{d}\lambda}{b(\lambda)}\right]. \tag{3.31}$$

(c) Near the end $\xi = 1$:

$$\chi(\xi) = \eta\left\{G_0 + \left[\frac{\int_0^1 [\mathrm{d}\lambda/b(\lambda)] \int_0^\lambda p(\phi)\,\mathrm{d}\phi}{b_0^{(1)} \int_0^1 [\mathrm{d}\lambda/b(\lambda)]} - \frac{1}{b_0^{(1)}} \int_0^1 p(\lambda)\,\mathrm{d}\lambda\right]\hat{\xi} + K_0 \exp\left(-\frac{b_0^{(1)}}{a_0^{(1)}}\hat{\xi}\right)\right\} \tag{3.32}$$

where $\hat{\xi} = (1-\xi)/\eta$.

A solution uniformly valid over the entire span is constructed by adding all three expansions and subtracting the common part which has cancelled through identically in the matching procedure. This common part is

$$CP = \frac{A_0}{b_0^{(0)}}\eta\tilde{\xi} + \eta C_0 + \left[\frac{A_0}{b_0^{(1)}} - \frac{1}{b_0^{(1)}} \int_0^1 p(\lambda)\,\mathrm{d}\lambda\right]\zeta\xi^+ + \eta G_0. \tag{3.33}$$

Thus, adding (3.30)–(3.32) all together and subtracting (3.33) yields

$$\chi(\xi) = \frac{\int_0^1 [\mathrm{d}\lambda/b(\lambda)] \int_0^\lambda p(\phi)\,\mathrm{d}\phi}{\int_0^1 [\mathrm{d}\lambda/b(\lambda)]} \int_0^\xi \frac{\mathrm{d}\lambda}{b(\lambda)} - \int_0^\xi \frac{\mathrm{d}\lambda}{b(\lambda)} \int_0^\lambda p(\phi)\,\mathrm{d}\phi$$
$$+\eta\left\{E_0 \exp\left(-\frac{b_0^{(0)}}{a_0^{(0)}}\frac{\xi}{\eta}\right) + K_0 \exp\left(-\frac{b_0^{(1)}}{a_0^{(1)}}\frac{1-\xi}{\eta}\right) + C_0 - \frac{C_0 - G_0}{\int_0^1 [\mathrm{d}\lambda/b(\lambda)]} \int_0^\xi \frac{\mathrm{d}\lambda}{b(\lambda)}\right\}. \tag{3.34}$$

The four constants C_0, G_0, E_0 and I_0 are determined through the usual boundary conditions at $\xi = 0$ and 1.

Note that in (3.34) all constants of integration arise in the first order term of the approximation. Away from the boundaries, the zeroth order solution which is comprised of the first two terms in (3.34) becomes dominant. This is recognized to be the solution of the equation

$$[b(\xi)\chi']' = -p(\xi)$$

with the boundary conditions $\chi(0) = \chi(1) = 0$, which corresponds to Saint Venant torsion of a thin rod with built-in ends. Thus, the very first approximation away from the boundaries is in fact the same for any type of boundary conditions.

It is readily recognized from the equations (3.30)–(3.34) that the first order approximation of the singular case involves only integrations over the coefficient $b(\xi)$, while $a(\xi)$ arises only in form of its boundary values. In other words, the first order approximation of (3.2) is the same for all shape functions $a(\xi)$ which have the same values at $\xi = 0$ and $\xi = 1$. A distinct approximation of (3.2) for different $a(\xi)$ is only obtainable when one treats with higher order perturbation. It is also easily identified that this remark does not hold for the regular case where the first approximation leads to a formulation which is dependent on the shape functions $a(\xi)$ and $b(\xi)$ over the entire interval $0 \le \xi \le 1$.

Having the general formulas (3.30)–(3.34), we are able to treat all special cases. If, for example, $b(\xi) = \text{const.} = 1$, it is the case of a bending member subjected to a constant normal force. If both a and b are constant, then we are led to a beam or shaft of constant cross section. For this case the differential equation (1.1) is easily integrated and the perturbation procedure seems to give no advantage except a little more insight to the propagation of errors in the numerical calculation when the cases $\varepsilon \to 0$, $\eta \to 0$ are considered.

4. APPLICATIONS

The two perturbation approaches are applied to analyze two different engineering problems with the torsion of a beam as an example for the singular case and the bending of a beam for the regular perturbation. The beam has non-uniform cross section and is clamped at both ends so it is constrained against warping in torsion near the ends and extension in bending.

(a) *The singular case*

The cross section we choose for the singular case is shown in Fig. 1. We consider a beam with parabolic variation of the height

$$h = h_0[1-\alpha\xi(1-\xi)]; \qquad \alpha < 4 \tag{4.1}$$

and with two built-in supports subjected to torque $t(x)$. The corresponding boundary conditions are

$$\phi = \phi' = \chi = \chi' = 0 \quad \text{at } \xi = 0, 1.$$

We omit all mathematical details, which are given in Ref. [5]. The final results for the constants of integration used in (3.30), (3.31) and (3.32) or (3.34) are

$$\begin{aligned} E_0 &= -C_0 = \frac{\int_0^1 [\mathrm{d}\lambda/b(\lambda)] \int_0^\lambda p(\phi)\,\mathrm{d}\phi}{[(b_0^{(0)})^2/a_0^{(0)}] \int_0^1 [\mathrm{d}\lambda/b(\lambda)]}; \\ K_0 &= -G_0 = \frac{-\int_0^1 [\mathrm{d}\lambda/b(\lambda)] \int_0^1 p(\phi)\,\mathrm{d}\phi}{[(b_0^{(1)})^2/a_0^{(1)}] \int_0^1 [\mathrm{d}\lambda/b(\lambda)]} + \frac{a_0^{(1)}}{(b_0^{(1)})^2} \int_0^1 p(\lambda)\,\mathrm{d}\lambda. \end{aligned} \tag{4.2}$$

These formulas give the twist and therefore together with the transformations (3.1) and the formulas (1.5) the inner forces.

Numerical results are computed for a constant specific torque and a realistic cross-section with the following data:

$$t_w = 0{\cdot}40\,\mathrm{m}, \qquad t_0 = 0{\cdot}20\,\mathrm{m}, \qquad t_u = 0{\cdot}10\,\mathrm{m}, \qquad l = 3{\cdot}50\,\mathrm{m}, \qquad h_0 = 2{\cdot}50\,\mathrm{m},$$
$$a(\xi) = b^3(\xi), \qquad b(\xi) = 1-\alpha\xi(1-\xi).$$

We have also evaluated the numerical value for η. Since this value is also a function of the length of the bar it is only worth to give its order of magnitude for realistic beams. For closed box-cross-sections in concrete as shown schematically in Fig. 1, we have found that

$$\eta \leq 0{\cdot}05.$$

This shows that one has in fact for such cases a real boundary layer problem.

Figure 2 shows the normalized warping moment T_ω as defined by equation (1.5) in the neighborhood of the boundary for several different choices of η in case of constant cross-section. Figure 3 shows the normalized Saint Venant torque T_s under the same conditions.

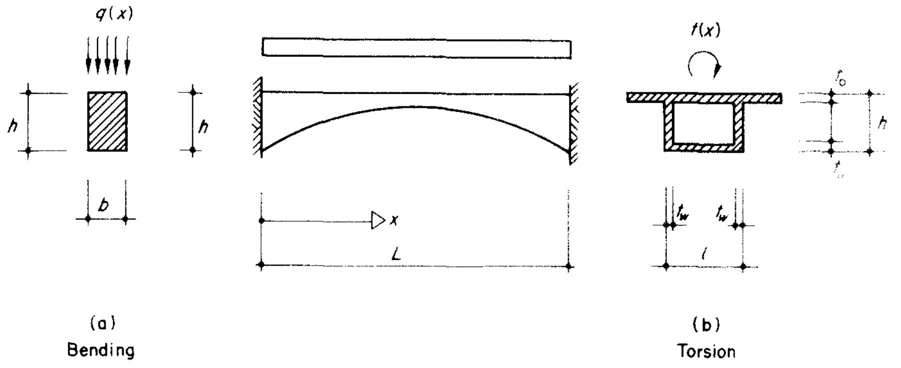

FIG. 1.

It is seen from these two graphs, which show one fifth and one half of the entire length of the bar, that the curves for different η are distinct appreciably near the boundaries but "converge" in the middle of the bar ($\xi = 0{\cdot}5$) to a common curve. This limit curve is in our example a straight line and as it easily can be verified, the solution of the reduced differential equation ($\eta = 0$).

The Figs. 4 and 5 demonstrate the influence of the variability of the shape functions characterized by the parameter α which describes the thickness in the middle of the beams ($\alpha = 1$ means constant cross-section, $\alpha = 4$ means no height at the middle cross-section). This influence to the Saint Venant torsional moment and to the warping moment is very little for values of the perturbational parameter $\eta < 0{\cdot}04$. Thus, one is in this case allowed to calculate with the simplified theory of constant cross-section (see Figs. 2 and 3) without loss of much accuracy. On the other hand, if $0{\cdot}04 < \eta < 1$, the distinction between the solution with constant and variable cross-sections becomes pronounced and one must use the general perturbation solution of variable cross-section.

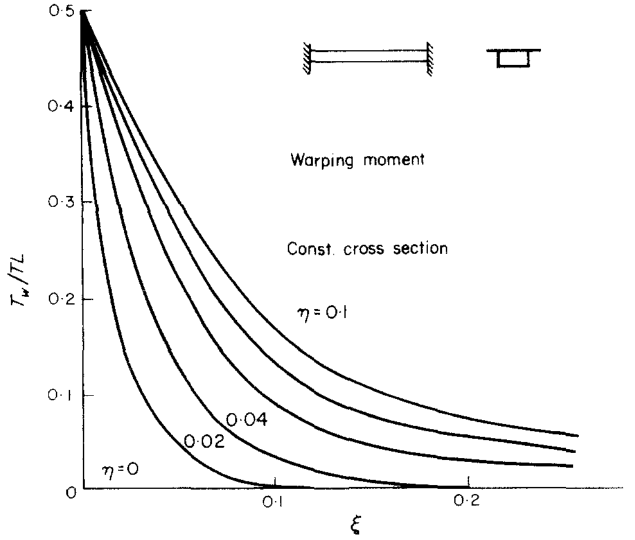

FIG. 2.

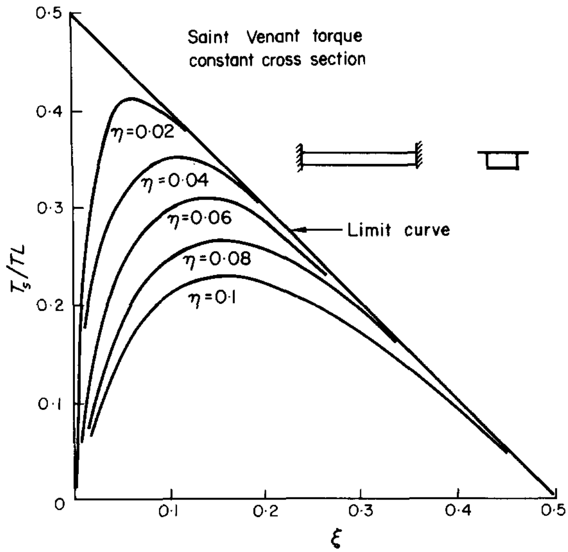

FIG. 3.

There remains still one question, namely the influence of an approximation of the shape function to the solution because the difference between the final solution for a constant cross-section and that for a variable cross-section becomes appreciable for $\eta > 0{\cdot}04$ and the torsional rigidity $GK(\xi)$ which enters into the perturbation calculation through the parameter $b(\xi)$ is in practice often given only pointwise. We have obtained in [5] that a replacement of a continuous shape function by a smooth approximation does affect the results only very little (percentage error <3 per cent) so that approximate representations of the shape functions are permissible.

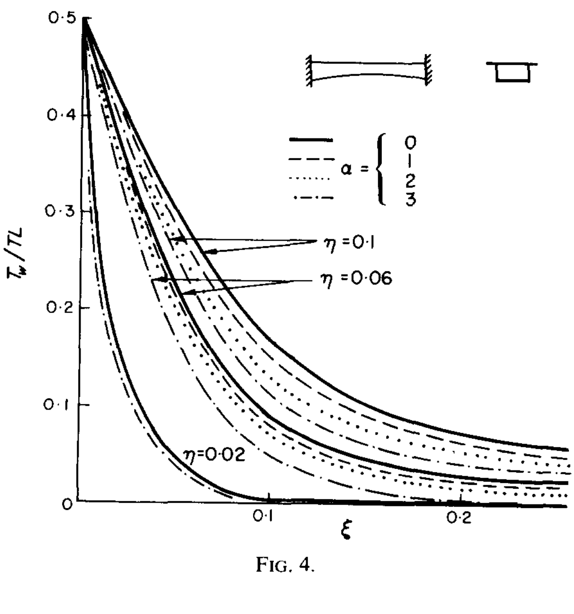

FIG. 4.

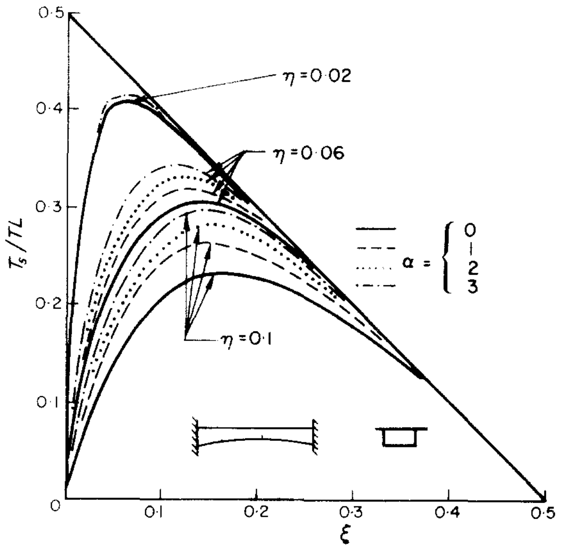

FIG. 5.

(b) *The regular case*

We restrict our considerations to a built-in bar in bending under a constant loading and assume a parabolic variation of the height as in (4.1). In addition, we assume a temperature variation, so that a normal force

$$N = -\omega \Delta T \,.\, E \,.\, b \,.\, h(\xi)$$

is induced. In the above ΔT denotes the temperature-raise relative to the unstressed state, ω the coefficient of thermal expansion, $bh(\xi)$ the area of the cross-section and E the modulus of elasticity. We emphasize that for $\Delta T \gtrless 0$ we have $\varepsilon \lessgtr 0$. We again omit the rather involved integrations and refer the interested reader to [5]. The constants of integration in (2.6) and (2.7) turn out to be

$$\left.\begin{aligned} C_0 &= 0; \qquad D_0 = 0; \\ A_0 &= -\int_0^{\frac{1}{2}} p(\lambda)\,\mathrm{d}\lambda = -\tfrac{1}{2}; \\ B_0 &= \frac{-A_0\int_0^{\frac{1}{2}}[\lambda\,\mathrm{d}\lambda/a(\lambda)]-\int_0^{\frac{1}{2}}[1/a(\lambda)]\int_0^{\lambda}(\lambda-\phi)p(\phi)\,\mathrm{d}\phi\,\mathrm{d}\lambda}{\int_0^{\frac{1}{2}}[\mathrm{d}\lambda/a(\lambda)]} \end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned} C_1 &= 0; \\ D_1 &= 0; \\ A_1 &= 0; \\ B_1 &= \frac{1}{\mathscr{L}_{03}(0{\cdot}5)}[-A_0 I'_1(0{\cdot}5)-B_0 I'_2(0{\cdot}5)-I'_3(0{\cdot}5)] \end{aligned}\right\} \tag{4.4}$$

where

$$I_1(\xi) = \int_0^\xi \frac{\xi-\lambda}{b(\lambda)^3}[1-\alpha\lambda+\alpha\lambda^2][\lambda\mathscr{L}_{13}(\lambda)-\mathscr{L}_{23}(\lambda)]\,\mathrm{d}\lambda \tag{4.5}$$

$$I_2(\xi) = \int_0^\xi \frac{\xi-\lambda}{b(\lambda)^3}[1-\alpha\lambda+\alpha\lambda^2][\lambda\mathscr{L}_{03}(\lambda)-\mathscr{L}_{13}(\lambda)]\,\mathrm{d}\lambda \tag{4.6}$$

$$I_3(\xi) = \int_0^\xi \frac{\xi-\lambda}{b(\lambda)^3}[1-\alpha\lambda+\alpha\lambda^2][\lambda\mathscr{L}_{23}(\lambda)-\mathscr{L}_{33}(\lambda)]\,\mathrm{d}\lambda \tag{4.7}$$

and

$$\mathscr{L}_{nm}(\xi) = \int_0^\xi \frac{\lambda^n\,\mathrm{d}\lambda}{(h/h_0)^m} \tag{4.8}$$

We have carried out numerical calculations for $\alpha = 1, 2$ and 3, which are shown in Fig. 6. This figure shows that the shape function is of great influence to the distribution of the bending moment. This is well known for the zero order moments M_0/PL^2 and is even true for the first order moments as Fig. 6 indicates.

Since $\alpha = 4$ belongs to a beam with zero thickness at $\xi = 0{\cdot}5$ we cannot obtain a particular curve for this case. The perturbation approach becomes singular [see e.g. equation (12)]. The curve $M_1/(PL^2)$ for $\alpha = 4$ would be shifting to ∞. This is consistent with the physical feature of the bending with nearly zero rigidity.

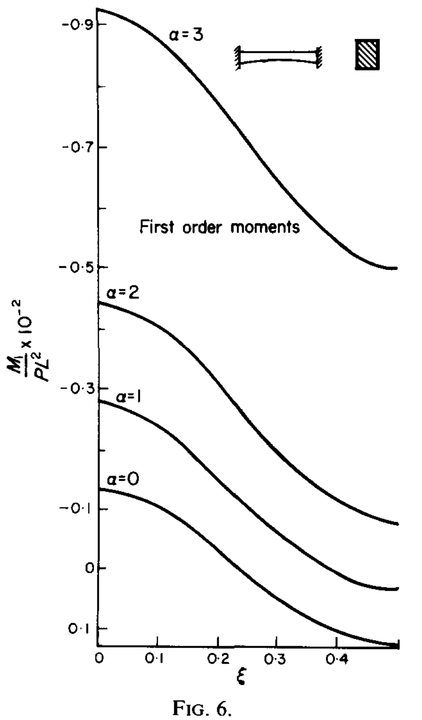

FIG. 6.

5. CONCLUSION

The application of the perturbation technique to the differential equation for warping-torsion or beam in bending under axial forces has led to the following results: in the regular perturbation case the elementary theory of pure warping or pure bending leads to an approximate answer with an error which is proportional to the perturbation parameter in the entire length of the beam. This however is not so in the singular perturbation case, where the approximate solution uniformly valid over the whole length of the beam is a composition of the outer expansion and inner solution. Here the perturbation parameter itself is not a measure for the error of the approximation. For the two structural problems we investigated, the singular perturbation analysis reveals that the theory of hollow shafts in torsion based on the pure Saint Venant assumption may be an oversimplification. The solutions are in error appreciably in the neighborhood of the boundaries. Similarly, the deformation of a large size cable under tension can not be treated by neglecting its bending rigidity. One should therefore carry out the analysis in all these cases with the exact theory, and the method of singular perturbation is the appropriate one to solve the equations arising from the exact theory.

As for the variability of the cross-sections, the numerical example shows that it is not necessary to carry through the calculations with the exact shape function for the cross section in the case of singular perturbation. This can be replaced by a simpler function which coincides with the exact one at a certain number of points. When the perturbation parameter η is less than 0·04, the variability of the cross section can be neglected.

In the case of the regular perturbation the calculations show that the solutions in general depend appreciably on the variability of the cross sections. This is well known in the case of bending for the zero order solution and was also proved to be true for the first order solution here. If therefore higher order solutions are considered they always have to be carried through without any simplifications as far as the variability of the cross sections is concerned.

One last point about the application of the singular perturbation method is worth while to mention. If the coefficient associated with the highest order term in the differential equation is small, difficulties in numerical calculation may arise if the common methods of integration are used. This is readily seen if the differential equation is solved when the coefficients are constant, for which the solution is obtained in a form which involves small numbers as divisors. The singular perturbation solution avoids this difficulty and is therefore—even for the equation with constant coefficients—the appropriate way of numerical approach.

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Zusammenfassung—Für den Biegezugstab und für das Problem der Wölbkrafttorsion wird eine Störungsrechnung durchgeführt und zwar unter den allgemeinsten Voraussetzungen variabler Koeffizienten, Belastung und beliebiger Randbedingungen. Für beide Fälle—den singulären und den regulären—werden die Störungsapproximationen explizite angegeben bis und mit dem ersten den Störparameter enthaltenden Term. Für den singulären Fall ist zudem eine über das ganze Integrationsintervall gültige uniforme Approximation angegeben. Zwei Beispiele illustrieren die Anwendung.

Абстракт—Дается решение методом возмущений для определяющего дифференциального уравнения, касающегося изгиба балки, нагруженной осевыми силами и депланации стержня при кручении. Ьолка или стержень могут иметь переменное поперечное сечение, переменную нагрузку и общие краевые условия. Для двух случаев, т.е. сингулярного и регулярнрго, получаются решения с помощью возмущений в явном виде, до первого члена возмущения/и заключая этот член/. Кроме того дается решение для сингулярного случая, равномернр важно по всей длине балки. Приводатся два примера, иллюстрирующие применения метода.